

# BEAM EXTRACTION AT A THIRD INTEGRAL RESONANCE IV

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In this report, we examine the curving of the separatrices caused by the fourth order term in the Hamiltonian, which we take to be

$$H = \left(\nu - \frac{m}{3}\right) \rho - A(2\rho)^{3/2} \cos 3\gamma + B(2\rho)^2, \quad (1)$$

where we have added to the Hamiltonian (I-1) the fourth order term derived in report III. The variables  $\rho, \gamma$  are the final transformed variables  $\underline{\rho}', \underline{\gamma}'$  of reports II and III, but they closely approximate the variables  $\rho, \underline{\gamma}$  of report I.

In Fig. 1, we sketch curves of constant  $H$ . The fixed points are solutions of the equations

$$\frac{\partial H}{\partial \rho} = \frac{\partial H}{\partial \underline{\gamma}} = 0, \quad (2)$$

whose solutions are

$$\begin{aligned} \underline{\gamma} = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ or } \frac{\pi}{3}, \pi, \frac{5\pi}{3}, \\ (2\rho)^{1/2} = \frac{3A}{8B} \pm \left[ \left( \frac{3A}{8B} \right)^2 + \frac{\frac{m}{3} - \nu}{B} \right]^{1/2} \text{ or} \\ - \frac{3A}{8B} \pm \left[ \left( \frac{3A}{8B} \right)^2 + \frac{\frac{m}{3} - \nu}{B} \right]^{1/2}. \end{aligned} \quad (3)$$

If  $v < \frac{m}{3}$ , there are two cases: In case (a),  $B > 0$ , there is one fixed point at each of the six angles  $\gamma$ , at a radius given by choosing the plus sign, since  $(2\rho)^{1/2}$  cannot be negative. In case (b),  $B < 0$ , there are two fixed points at each of the three angles  $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$ , given by choosing either sign in the appropriate formula for  $(2\rho)^{1/2}$ . For  $v > \frac{m}{3}$ , there are again two cases, but cases (a) and (b) occur for  $B < 0$  and  $B > 0$  respectively and the diagrams are rotated through an angle  $\pi/3$ . The radius to the unstable fixed points (corners of the triangle), when the coefficient  $B$  is very small is given approximately by

$$(2\rho)^{1/2} \doteq \left( \frac{\frac{m}{3} - v}{3A} \right) \left[ 1 - \frac{4B \left( \frac{m}{3} - v \right)}{9A^2} \right]. \quad (4)$$

The round bracket is the same result as in paper I, and the square bracket gives the reduction in radius to the corner of the triangle due to the  $B$ -term. The radius to the three extra stable fixed points is, for small  $B$

$$(2\rho)^{1/2} \doteq 3A/4B. \quad (5)$$

This gives a rough estimate of the amplitude at which curvature of the separatrix becomes important. In the notation introduced by Eq. (I-4), the fixed points lie at the radii

$$(2\rho)^{1/2} \doteq \left( \frac{2X_0}{\sqrt{3}} \right) [1 - 4\delta], \quad (6)$$

and

$$(2\rho)^{1/2} \doteq \frac{2X_0}{\sqrt{3}} \quad / (4\delta), \quad (7)$$

where

$$\delta = \frac{4BX_0^2}{m - 3v} \quad (8)$$

is a measure of the importance of the B-term.

For small B, the separatrix is the curve

$H = H_0 \doteq -4AX_0^3/3\sqrt{3}$ . It cuts the radius  $\underline{r} = 2\pi/3$  or  $\pi/3$  at radii which are solutions of the equation

$$\delta(2\rho)^2 \pm \frac{2}{\sqrt{3}} X_0 (2\rho)^{3/2} - \frac{3}{2} X_0^2 (2\rho) + \frac{3}{8} X_0^4 = 0. \quad (9)$$

The relevant solution is approximately

$$(2\rho)^{1/2} \doteq \frac{1}{3|\delta|} \frac{2X_0}{\sqrt{3}}, \quad (10)$$

which gives the radius to the farthest out point on the separatrix in either case.

Equation (I-6) for the separatrix now becomes, if we keep terms of order  $\delta$ ,

$$\begin{aligned} & (P - X_0/\sqrt{3})(P - \sqrt{3}X + 2X_0/\sqrt{3})(P + \sqrt{3}X + 2X_0/\sqrt{3}) \\ & - (3\sqrt{3}\delta/2X_0) \left[ (X^2 + P^2)^2 - 16X_0^4/9 \right] = 0. \end{aligned} \quad (11)$$

If we take the horizontal separatrix, and put  $P \doteq X_0/\sqrt{3}$  except in the first term, we find

$$P = (X_0/\sqrt{3}) \left[ 1 - 5\delta/2 - 3\delta X^2/2X_0^2 \right]. \quad (12)$$

This gives the first order deviation of the horizontal separatrix from a straight line, and also shows that the straight portion of the separatrix is raised or lowered slightly, depending on the sign of  $\delta$ .

As an example, in the case of a single sextupole, A is given by formula (III-14):

$$A = H_{33m} = \frac{eR\beta^{3/2}F}{24\pi M\gamma\omega}, \quad (13)$$

and B is given by formula (III-15):

$$\begin{aligned} B = H_{400} &= - \frac{4.89 \beta^2 e^2 R^2 F^2}{64\pi^2 M^2 \gamma^2 \omega^2} \\ &= - 44 A^2 / \beta. \end{aligned} \quad (14)$$

We substitute in Eq. (8) and use Eq. (I-5):

$$\delta = - \frac{44}{9\beta} \left( \frac{m}{3} - v \right) \quad (15)$$

With n homologous sextupoles, this is reduced by a factor  $n^2$  as noted in report III.

If there are octupole terms distributed according to

$$\begin{aligned} B_z &= G(\theta)(x^3 - 3xz^2), \\ B_x &= G(\theta)(3zx^2 - z^3), \end{aligned} \quad (16)$$

an analysis paralleling that in report II gives for the contribution to  $H_4 \ 0 \ 0$ ,

$$H_{4 \ 0 \ 0}^{\text{oct}} = \frac{3eR\beta^2 \bar{G}}{32M\gamma\omega}, \quad (17)$$

where  $\bar{G}$  is the average of  $G(\theta)$  around the circumference.

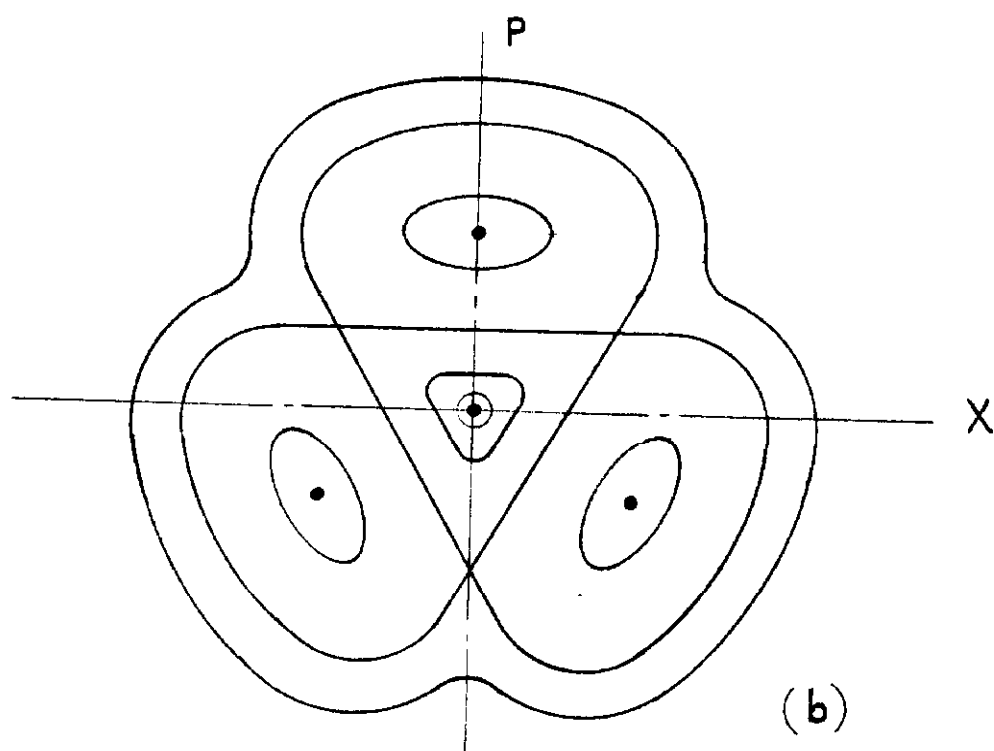
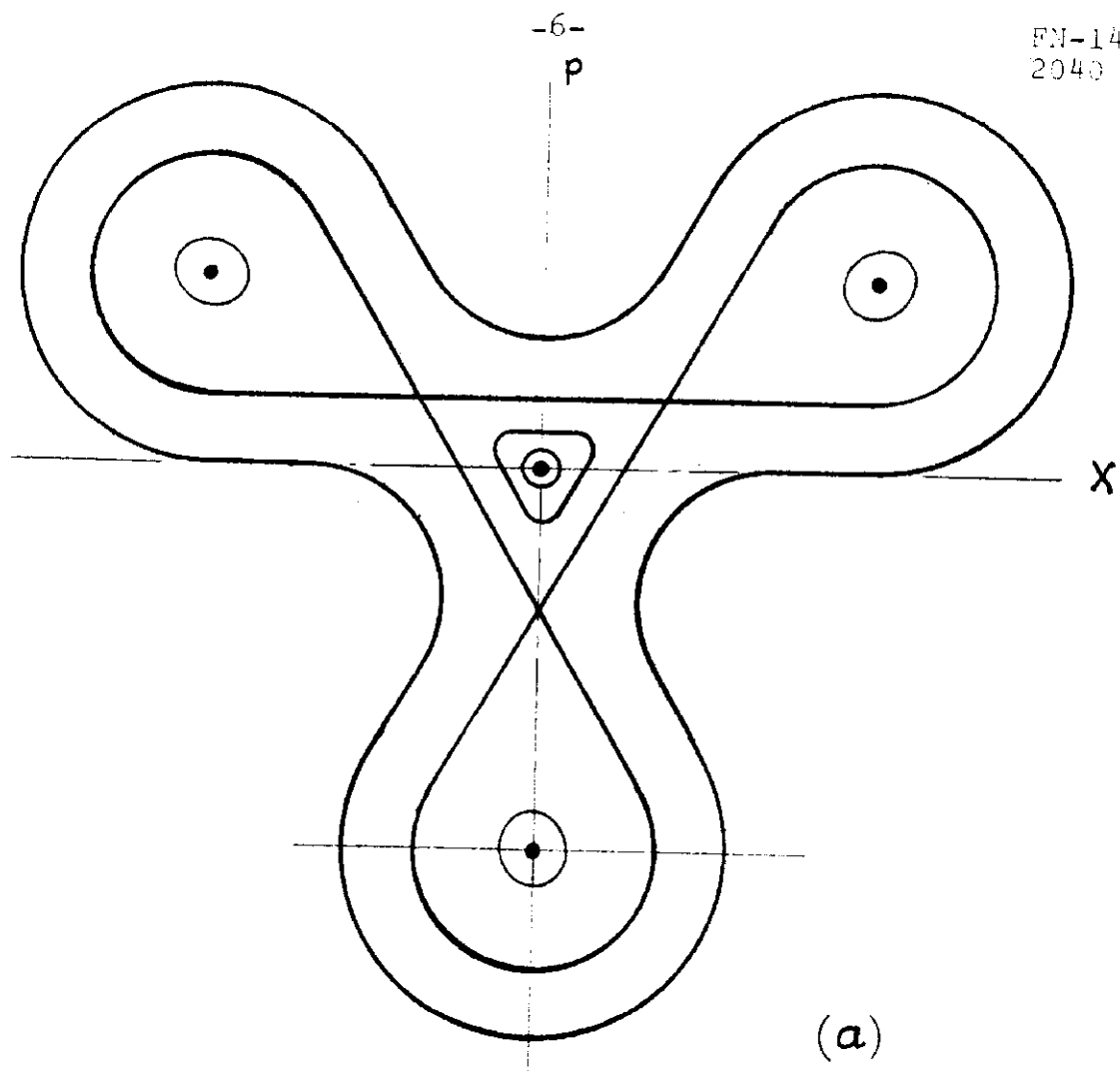


FIG.1 PHASE PLANE NEAR THIRD INTEGRAL RESONANCE